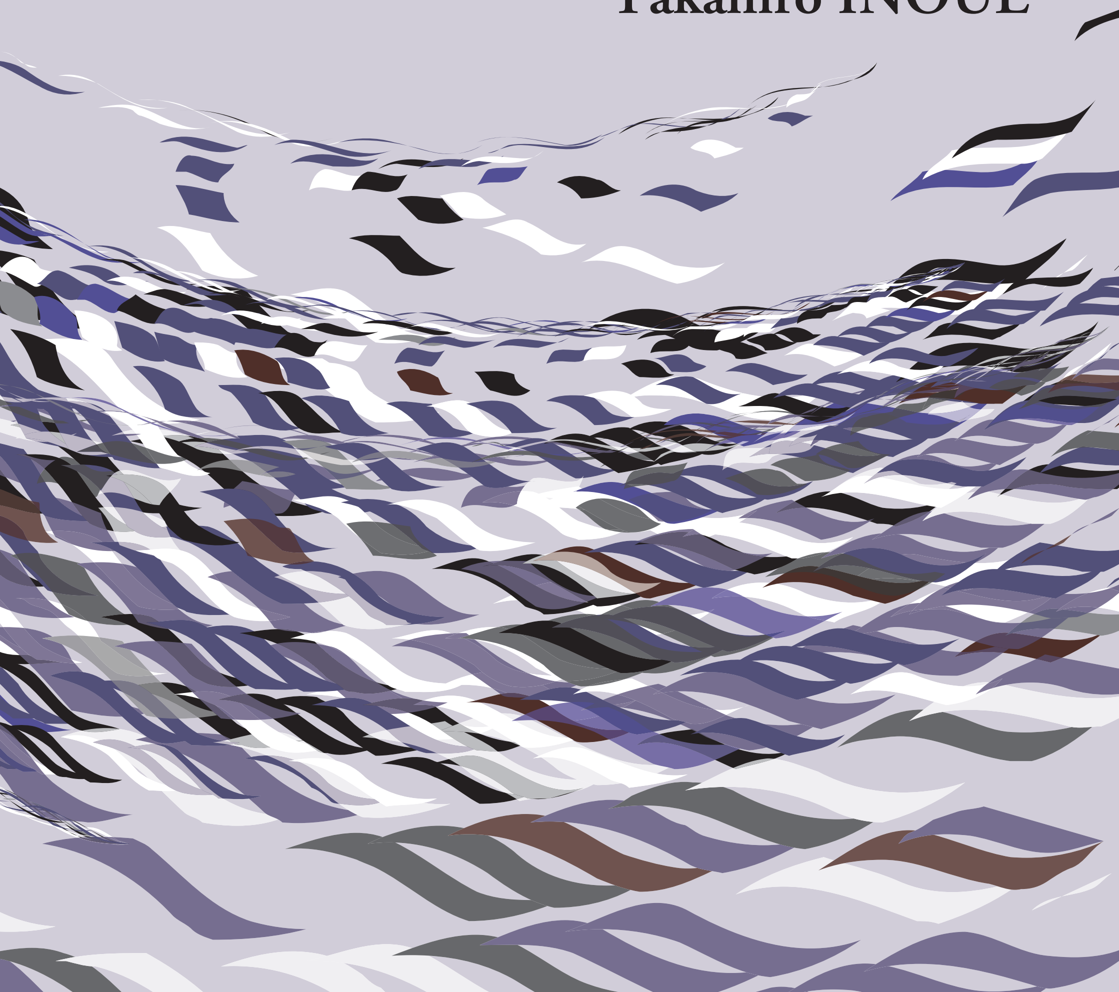
FRACTIONAL CALCULUS

—A NEW APPROACH FROM AN OPERATOR-BASED FORMULATION—

Third Edition

Takahiro INOUE



FRACTIONAL CALCULUS

—A NEW APPROACH FROM AN OPERATOR-BASED FORMULATION—

Third Edition

Takahiro INOUE

Professor Emeritus, Kumamoto University

**Dedicated to my wife Toshiko
and to my daughter and son.**

Preface to the First Edition

This book is an extended version of my previous books published under the title "A Unified Theory of Generalized Differentiation and Integration." The content of the previous books is extended and enhanced especially about the spaces on which the proposed theory can be constructed.

It is shown that the function space $\mathcal{F}$ which is introduced in my previous books can be extended to the generalized function space $\mathcal{F}_\infty$ by adding the delta function δ and its higher derivatives $p^n \circ \delta$ to $\mathcal{F}$. The subspace $\mathcal{N}_\infty$ which is the kernel of the semi p-operator p on $\mathcal{F}_\infty$ and the subspace $\mathcal{S}_{p,\infty}$ which is the image of $\mathcal{F}_\infty$ by p will play important roles in constructing the coset space $\mathbf{F}_\infty (= \mathcal{F}_\infty / \mathcal{N}_\infty)$. There is a homomorphism from $\mathcal{F}_\infty$ to this coset space under both addition $+$ and convolution $*$. A linear semi p-operator on this coset space can be induced from a linear semi p-operator on $\mathcal{F}_\infty$. In this extension, usual Leibniz's rule cannot be applied. It must be replaced with weighted Leibniz's rule with a constant weight $1/2$.

Improvement and refinement have been done in many places throughout this book. The motive and the aim of publishing this book are the same as those of the first edition of my previous book. So I will cite the preface of that book here.

> The aim of this book is to give a unified theory of generalized differentiation and integration.
>
> In this book, a differentiator-like operator to arbitrary order has been defined and a theory on fractional calculus has been developed from a new standpoint. It is well-known that classical differentiation and integration to arbitrary order have been well established in classical fractional calculus[1]-

[3] and in distribution theory[24],[25].

The theory presented in this book is constructed on a more general operator-based formulation which includes not only the previous formulations of fractional calculus but also a new formulation based on general differentiator-like linear operators to arbitrary order.

The incentive to develop this theory dates back to my master thesis. When I was a senior student in the undergraduate course of the Department of Electrical Engineering, Kumamoto University(KU), I began my theoretical research on a wave propagation problem so-called the Sommerfeld Problem under guidance of Prof. A. Yokoyama. At that time, Prof. Yokoyama developed a very efficient operational method to derive an asymptotic formula for a far-zone field of a dipole radiation effected by the plane earth. I was very much impressed by the beauty of his method and I tried to found his method on a more rigorous mathematical ground in my master thesis.

My main idea is to consider a sequence $\{\lambda_n\}_{n=0}^{\infty}$ generated by the equation $p^n \circ f = \lambda_n f$, where f and λ_n are functions and p is a differentiator-like operator. In my master thesis, I derived the main theorem in Chapter 3 of this book and applied it to the Sommerfeld Problem. When I was preparing my master thesis, I noticed that, since the Gamma function can be regarded as the interpolation of a factorial, it would be possible to extend differentiation and integration to arbitrary order by using the Gamma functions. So I dreamed to extend λ_n to non-integer index values in order to make it easier to evaluate the contour integrals around a pole or a branch point that we encounter in the Sommerfeld Problem.

After graduating from the Graduate School of Electrical Engineering in KU, I worked for Hitachi Ltd. in the field of an IC technology as a research staff. About four years later, I joined the Faculty of Engineering, KU, where I engaged in education and research on electronic circuits until my retirement. During these periods, I had been almost apart from my dream though I sometimes continued my private research on weekends. After my retirement from KU, I started again my theoretical work to pursue my dream.

At the end of 2014, I decided to write a book about my work on constructing a unified theory of generalized differentiation and integration by extending $p^n \circ f = \lambda_n f$ to arbitrary order. To tell the truth, I have been in complete ignorance of the fact that fractional calculus has already been well developed and it has a three-century history. I learned fractional calculus from a book written by Oldham and Spanier[1] recently. From January to May, 2015, I was so busy and nervous because I had to check whether the results given in Oldham & Spanier's book can be reproduced from my theory. Finishing this work, I convinced that my theory can reproduce the classical results and it is an extension of classical fractional calculus.

In this book, a differentiation operator in fractional calculus is generalized to a more general operator that I call semi p-operator which may be linear or non-linear. Then, we consider a sequence of functions $\{\lambda_n\}_{n=0}^{\infty}$ generated by the equation $p^n \circ f = \lambda_n f$ where p is a linear semi p-operator and f is a function. And we introduce its integral representation in a complex region. Like the Gamma function which is an extension of a factorial, we can extend λ_n to arbitrary index values via its integral representation. Thereby we can define $p^z \circ f = \lambda_z f$ where z is an arbitrary order. The classical fractional calculus can be derived as a special case and various special functions of arbitrary order follow from λ_z by choosing different pairs of p and f. Some applications of the theory are given in the last chapter with special emphasis on the electromagnetic wave problems.

Chapter 1 in this book is a brief introduction of the history of fractional calculus.

The following Chapter 2 is the preliminaries which give the definitions of a function space and operators to be used in the succeeding chapters. In particular, Definition 2.1 gives the definition of a function space $\mathcal{F}$ on which the present theory is constructed. Remark 2.14 shows that we can obtain weighted Leibniz's theorem by extending usual Leibniz's theorem when the weight is a constant and the multiplication operation is the binomial operation $\odot$ defined in Definition 2.13. Remark 2.15 shows the relationship between the λ-sequence and the Riccati differential equation. The function space $\mathcal{F}_\infty$ is introduced in Section 2.2 and its relationship to $\mathcal{F}$ is clarified there in detail. Section 2.3 gives interesting formulae about commutators.

In Chapter 3, the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is introduced and the fundamental theorems on this sequence are proved. In addition, Theorem 3.15 gives the derivation of λ_m for when a semi p-operator is a sum of n different semi p-operators and it is applied to a product of n different functions in $\mathcal{F}$.

In Chapter 4, the more general sequences of functions $\{\mu_n\}_{n=0}^{\infty}$'s and their induced sequences are considered and discussed in three interesting cases. It is proved that those induced sequences have expansions of a similar form as λ_n has.

The integral representation of λ_n and its extension to arbitrary order are described in Chapter 5. In this chapter, the relations to the previous definitions of fractional calculus are discussed. Other interesting results shown in this chapter are as follows. Theorem 5.9 presents the relation that holds between the generating function of λ_n and its partial sum. Remark 5.13 shows that Riemann's zeta function $\zeta(\beta+1)$ can be obtained by the β-fold fractional integration of $x/(e^x-1)$ when evaluated at $x=0$. Theorem 5.14 shows the shift-invariance property of the operator $(d/dz)^\alpha$ where α is a complex order. Theorem 5.18 gives a weaker sufficient condition for $(d/dx)^\alpha$ to be a continuous operator to bounded regular functions. Case I in Section 5.7 shows that the extension of $(\gamma D)^n \circ f = \lambda_n f$ to arbitrary order can be derived from its induced sequence.

In Chapter 6, various special functions are derived as λ_n for different pairs of p and f, and they are extended to arbitrary order. One of the interesting results in this chapter is Theorem 6.6 which shows that the composition rule holds for the operator $(d/dz)^\alpha$ when it is applied to a complex function which is regular in an open disk in the complex plane and satisfies some additional conditions.

Some applications of the theory are given in Chapter 7 where an order-reduction of ordinary differential equations of a certain type, solutions of basic differ-integral equations, an infinite continued square root, diffusion-type equations, a semi-infinite electric circuit, and electromagnetic wave problems are investigated. Theorem 7.1 supplemented by Remarks 7.1-7.2 gives a solution of a basic differ-integral equation. Section 7.3 gives an interesting example of the infinite continued square root that is derived as an application of Remark 2.15. Subsection 7.4.3 shows that the quasi-classical approximation(the WKB approximation) for a solution of Schrödinger's equation in quantum mechanics can be easily obtained by using λ_n. The Sommerfeld problem is treated in

Subsection 7.4.5.

Theorems A.1 and A.2 in Appendix A give the important theorems which are used in Sections 5.4 and 5.7 and in Chapters 6-7.

Finally, I should like to express my sincere gratitude to Prof. Akira Yokoyama for guiding me to this fruitful field of science and I must also express my thanks to my wife Toshiko for her patience and continuous support to this work.

Takahiro Inoue

Kumamoto,
March, 2017

Preface to the Second Edition

This book is intended to formulate fractional calculus by using an operator-based method. The differentiation operator in classical fractional calculus is extended to a more general linear operator which satisfies Leibniz's rule. The key idea is to consider a sequence of functions produced by iterated applications of such a linear operator to a function belonging to some specified function space. Fractional power of such a linear operator is defined by extending the integral representation of a function in such a sequence by analytic continuation. The advantage of this theory is that it can grasp special functions within its scope and this scope is broad enough to induce a theory of fractional calculus on a coset space.

The contents newly added in the second edition are Remarks 2.17-2.18, Remark 3.1, Theorem 3.2, Remark 3.2, Remark 3.4, Definition 3.3, Theorems 3.7-3.10, Remark 3.6, Definition 5.2, Theorem 5.4, Remark 5.6, Theorem 5.5, Remark 5.10, Theorem 6.4, and Remarks 6.4-6.5.

Remark 2.17 shows that $\mathcal{F}_\infty$ in Remark 2.16 is an algebra, and $\mathcal{N}_\infty$ and $\mathcal{S}_{p,\infty}$ are ideals in this algebra. In Remark 2.18 (and also in Theorem 2.15), it is shown that when $\mathcal{F}_\infty$ is defined by Remark 2.16, the semi p-operator $\hat{p}$ on $\mathbf{F}_\infty$ which is induced from the semi p-operator p on $\mathcal{F}_\infty$ behaves in accordance with p on $\mathcal{F}_\infty$.

Remark 3.1 gives the λ-expansion of λ_{n+1} in the function space $\mathcal{F}$ when λ_1 commutes with a linear semi p-operator p. Theorem 3.2 gives the generating formula of λ_n in $\mathcal{F}$ for when λ_1 and p commute. Remark 3.2 shows how the generating function of λ_{n+1} in the function space $\mathcal{F}$ is obtained from that of λ_n in $\mathcal{F}$ and that the generating function of λ_{n+1} in $\mathcal{F}$ is given by $\lambda_1(x+t)e^{\lambda_1(x)t}$ when p and λ_1 commute. In Remark 3.4, it is shown that $p \circ f^n = f^{n-1}(p \circ f) = f^n(p \circ 1) = 0$ holds for $n = 1, 2, \ldots$ when p commutes with $f(\in \mathcal{F})$. Definition 3.3 gives the definition of λ_n in $\mathcal{F}_\infty$ of Remark 2.16. The generating formula of λ_{n+1} in such an $\mathcal{F}_\infty$ is given in Theorem 3.7. Theorem 3.8 gives the alternative generating formula of λ_n in such an $\mathcal{F}_\infty$. Theorem 3.9 gives the λ-expansion of λ_{n+1} for when the function space $\mathcal{F}_\infty$ and the linear semi p-operator p are defined by Remark 2.16 and λ_n is defined by Definition 3.3. The generating function of λ_{n+1} in the function space $\mathcal{F}_\infty$ of Remark 2.16 is given by Theorem 3.10. Remark 3.6 gives the alternative generating function of λ_n in $\mathcal{F}_\infty$ of Remark 2.16. In this remark the alternative expression of λ_n in Theorem 3.8 is used.

Definition 5.2 gives the definition of $p^z \circ c$ where c is a constant function. Theorem 5.4 gives the integral representation of λ_{n+1} in $\mathcal{F}_\infty$ of Remark 2.16. Its extension to arbitrary complex order is given in Remark 5.6 along with the integral representation of λ_n in such an $\mathcal{F}_\infty$ and its extension to complex order. In Theorem 5.5, it is shown that when $\mathcal{F}_\infty$ is defined by Remark 2.16, the definition of λ_{z+1} introduced in Remark 5.6 is equivalent to that to be introduced in Remark 5.10. In Remark 5.10, the definition of λ_{z+1} in $\mathcal{F}_\infty$ of Remark 2.16 is given by using the generalized binomial expansion.

Theorem 6.4 gives the theorem on how $(d/dx)^z \circ x^m$ can be defined for when $m = -1, -2, \dots$ and $z \neq m, m-1, m-2, \dots$. Remark 6.4 shows that the definition of $(d/dx)^z \circ x^m$ given in Theorem 6.4 is broader than that given in Theorem 6.3 when $m = -1, -2, \dots$. In Remark 6.5, it is shown that $(xd/dx)^z \circ x^{-1} = (-1)^z x^{-1}$ holds even for $\Re z < 0$ in addition to $z = 0, 1, 2, \dots$.

Small modifications and corrections have been made throughout this book to enhance the clarity of each assertion.

Lastly, it is my small hope that this book will be some contribution to the development of fractional calculus.

Takahiro Inoue

Kumamoto,
August, 2017

Preface to the Third Edition

The contents newly added in the third edition are Remark 2.4, Remark 2.11, Definition 5.5, Theorem 5.20, Remark 5.16, Theorem 5.21, Remark 5.17, Definition 5.6, Theorems 5.22-5.23, and Remark 5.18.

Remark 2.4 calls readers' attention to that L_0 in the proof of Theorem 2.1 is assumed to be a linear operator. So L_0 must be a homogeneous operator in addition to that it is a homomorphic mapping with respect to addition. In this remark, the assumption that L_0 is a homogeneous operator is justified if we define $p \circ f^\alpha$ for $f > 0$ by $p \circ f^\alpha = \alpha f^{\alpha-1} p \circ f$ where α is an arbitrary real number. Remark 2.11 shows the alternative statement of Definition 2.2 in terms of λ_1's of $f, g \in \mathcal{F}$ under some restricted condition on f and g.

Definition 5.5 gives the definition of a uniformly convergent infinite series of polynomials in $f(\in \mathcal{F})$ in a generalized sense with respect to $f(x) \in \mathcal{D}$ where $\mathcal{D}$ is a simply-connected region in the complex plane. Under the condition that a linear semi p-operator is applicable term-by-term to any such a uniformly convergent infinite series of polynomials in $f(\in \mathcal{F})$, Theorem 5.20 gives an explicit formula of $p \circ F(f)$ for a composite function $F(f)$ when $F(z)$ is a regular function of z in $\mathcal{D}$. Remark 5.16 calls readers' attention to that the result in Theorem 5.20 corresponds to the chain rule for differentiation of a composite function in classical calculus. Under some condition of p^α similar to that of p in Theorem 5.20, Theorem 5.21 gives the integral representation of $p^\alpha \circ F(f)$ for a complex order α if the existence of the integral representation of $p^\alpha \circ f^k$ is assured for each non-negative integer k. Remark 5.17 derives the explicit algebraic formula for $p^n \circ F(f)$ which corresponds to Faà di Bruno's formula for the n-th order differentiation of a composite function. And, in addition, it shows that the expression for $p \circ F(f)$ given in Theorem 5.20 is justified also by Theorem 5.21.

Definition 5.6 gives the definition of the Laplace transform of $p^\alpha \circ f$ where α is a complex order. Theorem 5.22 shows that a classical result for the Laplace transform $\mathcal{L}[(d/dx)^n \circ f]$ can be obtained from Definition 5.6 when the order n is a positive integer. Theorem 5.23 gives the formula for the Laplace transform $\mathcal{L}[(d/dx)^\alpha \circ f]$ when α is a real order. In Remark 5.18, the Laplace transform of $\{(ax+b)(d/dx)\}^\alpha \circ (ax+b)$ is derived by using Definition 5.6 when $a(\neq 0)$ and b are constants and α is an arbitrary complex order.

In this book, small corrections and improvements have been made in the places where I thought the assertion should be refined.

Takahiro Inoue

Kumamoto,
January, 2018

Contents

Chapter 1

Introduction

The interpolation of differentiation and integration to non-integer order is often called "*fractional differ-integration*" and the calculus based on this is called "*fractional calculus*"[1]-[3]. Since this interpolation can be extended to arbitrary complex order by analytic continuation, the theory on fractional calculus is the generalization and unification of differentiation and integration.

The history of fractional calculus can be traced back to Leibniz's response to L'Hospital's letter dated 30th September, 1695[4]. L'Hospital asked him what would result if $n = 1/2$ for $d^n f(x)/dx^n$. Leibniz responded prophetically that one day useful consequences will be drawn from it[1],[2]. In 1819, S. F. Lacroix[5] obtained

$$(\frac{d}{dx})^{\frac{1}{2}} x = \frac{2\sqrt{x}}{\sqrt{\pi}} \tag{1.1}$$

by extending integers m and n of

$$(\frac{d}{dx})^n x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-n)} \cdot x^{m-n} \tag{1.2}$$

to $m = 1$ and $n = 1/2$, where $\Gamma(\cdot)$ is the Gamma function. In 1822, J. B. J. Fourier[6] generalized differentiation and integration by regarding n of the formula

$$(\frac{d}{dx})^n f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} \beta^n \cos(\beta x - \beta\alpha + \frac{n\pi}{2}) d\beta \tag{1.3}$$

as any real number. In 1823, N. H. Abel[7],[8] solved the tautochrone problem by using fractional integration of order 1/2. From 1832 to 1855, J. Liouville[9]-[11] carried out major studies on fractional calculus. He expanded functions in series of exponential functions and defined the non-integer order derivatives by term-by-term fractional operations. He also discussed fractional derivatives as the limits of a discretized-version definition of fractional derivatives. In 1847, B. Riemann[12] derived the definition of fractional integration:

$$(\frac{d}{dx})^{-q} f(x) = \frac{1}{\Gamma(q)} \int_c^x (x-y)^{q-1} f(y) dy, \tag{1.4}$$

where $q > 0$. In 1884, H. Laurent[13] generalized Cauchy's integral formula (or Goursat's formula) for the n-th derivative of a complex function by extending Letnikov's work in 1872[14]. The unified discretized-version of fractional differentiation and integration was given by A. K. Grünwald in 1867[15] and extended by E. L. Post in 1930[16]. This unified discretized-version of fractional differentiation and integration is proven to be equivalent to the continuous version given by the Riemann-Liouville definition:

$$(\frac{d}{dx})^{q} f(x) = \frac{1}{\Gamma(n-q)} (\frac{d}{dx})^n \int_a^x (x-y)^{-q+n-1} f(y) dy, \tag{1.5}$$

where n is an integer such that $(n-1) \leq q < n$[2]. In 1899, O. Heaviside used fractional differentiation in his transmission line theory[17]. Though the rigorousness of the operational calculus developed by him had been criticized, it was later founded on a rigorous mathematical ground by J. Mikusiński[18]. Readers will find the further detailed history and the information on recent developments in References [1]-[3].

Fractional calculus provides us with a very powerful mathematical tool in analyzing natural phenomena in the real world since many natural phenomena we encounter in the real world are governed by the power law. The usefulness of fractional calculus is in that (i) the Fourier transform of a power law distribution of the form $|t|^{-\alpha}$ (where $0 < \alpha < 1$) in the time domain (e.g., a long-tailed decay) is also a power law of the form $\omega^{\alpha-1}$ in the frequency domain and (ii) the fractional differ-integration by time is directly related to the fractional power law in the frequency domain by the inverse Fourier transform. These properties are applied to the spectral analysis of relaxation processes and to the stochastic process of the fractional Brownian motion[19],[20]. Interesting discussions on these matters can be found in References [2],[20].

Another important fact regarding fractional calculus is that differentiation to non-integer order is based on a non-local property of a function to be differentiated whereas differentiation to integer order is based on a local property of the function. This is because fractional differentiation to non-integer order is essentially the interpolation given by an integral representation which depends on a non-local property of the integrand.

In the following chapters of this book, a differential operator is extended to a more general differentiator-like operator which I call "*semi p-operator*" in this book. In case where this semi p-operator is linear, the integer-order power of this operator is extended to arbitrary order with respect to a function belonging to some sufficiently broad function space. In this theory, the sequence which I call the "λ-*sequence*" in this book and its contour integral representation are essential. By using these concepts, the relation between the proposed definition and the classical definitions of fractional differ-integration is explained. In later chapters, applications to special functions and to some typical problems in physics are also presented.

Chapter 2

Preliminaries

2.1 Function Space $\mathcal{F}$ and Related Operators

Let X and Y be topological spaces. In addition, we assume that Y has the zero element 0_Y and the multiplicative identity element 1_Y under addition and multiplication in Y, respectively.

Throughout this book, 0_Y and 1_Y will be simply denoted by 0 and 1, respectively, wherever they will not be confused with the zero function 0 and the identity function 1, both of which are the elements of the following function space $\mathcal{F}$.

Definition 2.1 *A "function space $\mathcal{F}$" is a collection of functions $f : X \to Y$ such that*

(i) *$\mathcal{F}$ is a linear space;*

(ii) *$\mathcal{F}$ is closed under multiplication:$fg \in \mathcal{F}$ for any $f, g \in \mathcal{F}$;*

(iii) *Multiplication is commutative:$fg = gf$ for any $f, g \in \mathcal{F}$;*

(iv) *Multiplication is associative:$(fg)h = f(gh)$ for any $f, g, h \in \mathcal{F}$;*

(v) *Scalar multiplication is defined such that $(\alpha f)(\beta g) = (\alpha\beta)(fg) \in \mathcal{F}$ for any scalars α and β;*

(vi) *Addition and multiplication in $\mathcal{F}$ are connected by the distributive law: $(f + g)h = fh + gh$ for any $f, g, h \in \mathcal{F}$;*

(vii) *$\mathcal{F}$ has the identity function 1 such that $1f = f1 = f$ for any $f \in \mathcal{F}$;*

(viii) *For some $f \in \mathcal{F}$ such that $f \neq \alpha 1$ for any scalar α, there exists a unique $\tilde{f} \in \mathcal{F}$ such that $\tilde{f}f = f\tilde{f} = 1$.*

Note here that $fg(\in \mathcal{F})$ in (**ii**) *is not a composition of two mappings.*

In addition, we assume $\mathcal{F}$ has at least one (semi) p-operator defined below.

Definition 2.2 *A "semi p-operator" p is a mapping of $\mathcal{F}$ into itself such that*
(**i**) *for $c, 0 \in \mathcal{F}$, $p \circ c = 0$ if c is a constant function* † *;*
(**ii**) *for any $f, g \in \mathcal{F}$, $p \circ (fg) = (p \circ f)g + f(p \circ g)$ (Leibniz's rule).*
In case where p can be defined such that, under a certain definite rule, a unique $f \in \mathcal{F}$ is specified corresponding to each $g(= p \circ f) \in \mathcal{F}$, I call it "the p-operator of $\mathcal{F}$."
(Note here that $p \circ 1 = 0$ can be derived only from (**ii**) of Definition 2.2 if some $f \in \mathcal{F}$ has $\tilde{f}$. Hence, in Definition 2.2, (**i**) can be replaced with $[p, c] = 0$ if some $f \in \mathcal{F}$ has $\tilde{f}$ where $[\cdot, \cdot]$ is a commutator defined in Sec. 2.3.)

Definition 2.3 *If $p \circ f_n \to p \circ f$ as $f_n \to f$ for any $f_n, f \in S(\subset \mathcal{F})$, then p is called a continuous (semi) p-operator on S.*

Definition 2.4 *Let us call a subset $\mathcal{S}$ of $\mathcal{F}$ "a linear subspace of $\mathcal{F}$" when it is a linear space. And let us call a subset $\mathcal{S}$ of $\mathcal{F}$ "a subspace of $\mathcal{F}$" when it satisfies all the conditions in Definitions 2.1 and 2.2 of a function space $\mathcal{F}$. Note that if $\mathcal{S}$ is a subspace of $\mathcal{F}$, then it is a linear subspace of $\mathcal{F}$.*

Remark 2.1 Let us define df for $f \in \mathcal{F}$ by $df \stackrel{\Delta}{=} Hf - fH$, where H is an operator such that $H : \mathcal{F} \to \mathcal{F}$ and $Hc = cH$ if c is a constant function. Then, the operator d is a semi p-operator.

Remark 2.2 An example of a semi p-operator in tensor analysis is the covariant derivative under which the fundamental tensor and the unit tensor are constant functions.

Remark 2.3 Let h be such that $h \in \mathcal{F}$. And let p be a semi p-operator of $\mathcal{F}$. Then, it is easy to verify that hp is also a semi p-operator of $\mathcal{F}$.

Theorem 2.1 *Let $\mathcal{F}$ be the set of all infinitely smooth real functions $f : X \to Y$ where both X and Y are in $(-\infty, +\infty)$. And let p be a semi*

† Some appropriate subset of $\mathcal{F}$ can be chosen as the set of all constant functions in $\mathcal{F}$, say $\mathcal{S}_c$. Then the semi p-operator p of $\mathcal{F}$ is defined with respect to $\mathcal{S}_c$ by Definition 2.2.

p-operator of $\mathcal{F}$. Then, for all $x \in X$ such that $f(x) \neq 0$, $p \circ f$ can be expressed as

$$p \circ f = (fL) \circ (\frac{d}{dx}) \circ \ln |f|, \tag{2.1}$$

where L is a linear operator of $\mathcal{F}$ (see Definition 2.5).

Proof: By (**ii**) in Definition 2.2, we have

$$\frac{p \circ (fg)}{fg} = \frac{p \circ f}{f} + \frac{p \circ g}{g} \tag{2.2}$$

for all $x \in X$ such that $f(x) \neq 0$ and $g(x) \neq 0$ where $f, g \in \mathcal{F}$. Since (2.2) implies the logarithmic operation for the product of functions, there exists some linear operator $L_0 : \mathcal{F} \to \mathcal{F}$ such that

$$\frac{p \circ f}{f} = L_0 \circ \ln |f|. \tag{2.3}$$

In addition, to satisfy (**i**) in Definition 2.2, the linear operator L_0 can be written as

$$L_0 = L \circ (\frac{d}{dx}), \tag{2.4}$$

where L is a linear operator of $\mathcal{F}$. Q.E.D.

Remark 2.4 Note that L_0 in the proof of Theorem 2.1 is assumed to be a homogeneous operator in addition to being a homomorphic mapping under addition since L_0 is assumed to be a linear operator. Namely, $L_0\alpha = \alpha L_0$ is assumed to hold for any real number α. Such an assumption for L_0 is justified if we define $p \circ f^\alpha$ for $f > 0$ by $p \circ f^\alpha = \alpha f^{\alpha-1} p \circ f$ in accordance with (3.9) in Lemma 3.1. This is because when (2.3) holds, one can prove that for $f > 0$ and any real number α, $(p \circ f^\alpha)/f^\alpha = \alpha\{(p \circ f)/f\} \Leftrightarrow L_0\alpha = \alpha L_0$.

Remark 2.5 Let $\mathcal{F}$ be the set of all regular functions $f(z)$'s defined on a region $\mathcal{D}$ in the complex plane $\mathcal{C}$. And let p be a semi p-operator of $\mathcal{F}$. Then, for all $z \in \mathcal{D}$ such that $f(z) \neq 0$, $p \circ f$ can be expressed as

$$p \circ f = (fL) \circ (\frac{d}{dz}) \circ \ln f, \tag{2.5}$$

where L is a linear operator of $\mathcal{F}$.
Note here that for $f : X \to Y$, $X = D$ by its relative topology to the topology of $\mathcal{C}$ and $Y = \mathcal{C}$.

Remark 2.6 Let $\mathcal{F}$ be the set of all infinitely smooth real functions $f : X \to Y$ where both X and Y are in $(-\infty, +\infty)$. For $f \in \mathcal{S} \equiv \{f : f \in \mathcal{F}, f(x) \neq 0 \text{ for all } x \in X\}$, let us define s_n by

$$s_n \circ f = f(x)(\frac{d}{dx})^n \ln |f(x)|,$$

where n is a positive integer. Then, s_n is an operator $s_n : \mathcal{S} \to \mathcal{F}$. Note that if $n = 1$, then the operator s_1 can be extended to a linear semi p-operator of $\mathcal{F}$ since $f(x)(d/dx) \ln |f(x)| = (d/dx)f(x)$. If $n = 2$, then s_2 is a non-linear operator since

$$f(x)(\frac{d}{dx})^2 \ln |f(x)| = (\frac{d}{dx})^2 f(x) - \frac{1}{f(x)}\{\frac{df(x)}{dx}\}^2.$$

Lemma 2.1 *Let p be a semi p-operator of $\mathcal{F}$. If $f \in \mathcal{F}$ and $\tilde{f}$ exists in $\mathcal{F}$, then we have*

$$p \circ \tilde{f} = -\tilde{f}^2 (p \circ f).$$

Proof: By (**ii**) in Definition 2.2, we have

$$p \circ (\tilde{f} f) = (p \circ \tilde{f}) f + \tilde{f}(p \circ f).$$

By (**i**) in Definition 2.2. the left-hand side of the above equation is zero since $\tilde{f} f$ is a constant function by (**viii**) in Definition 2.1. Q.E.D.

Definition 2.5

(**i**) *If L is a mapping such that $L \circ f \in \mathcal{F}$ for any $f \in \mathcal{F}$, then L is called an operator of $\mathcal{F}$. (Note that any $g \in \mathcal{F}$ is an operator of $\mathcal{F}$ because $gf \in \mathcal{F}$ for any $f \in \mathcal{F}$. Hence we can write $gf = g \circ f$.);*

(**ii**) $(Lf) \circ g \triangleq L \circ (fg)$ *for any $f, g \in \mathcal{F}$;*

(**iii**) $(fL) \circ g \triangleq f(L \circ g)$ *for any $f, g \in \mathcal{F}$;*

(**iv**) *If an operator L of $\mathcal{F}$ satisfies $L \circ (\alpha f + \beta g) = \alpha L \circ f + \beta L \circ g$ for any $f, g \in \mathcal{F}$ and any scalars α, β, then L is called a linear operator of $\mathcal{F}$.*

Definition 2.6 *Multiplication of two operators L_1 and L_2 of $\mathcal{F}$ is defined by*

$$(L_1 \circ L_2) \circ f \triangleq L_1 \circ (L_2 \circ f)$$

for all $f \in \mathcal{F}$. (Note that (**ii**) *and* (**iii**) *in Definition 2.5 can be removed under Definition 2.6 since any $g \in \mathcal{F}$ can be regarded as an operator of $\mathcal{F}$.)*

Definition 2.7 *Addition and subtraction of two operators L_1 and L_2 of $\mathcal{F}$ are defined by*

$$(L_1 \pm L_2) \circ f \triangleq L_1 \circ f \pm L_2 \circ f$$

for all $f \in \mathcal{F}$.

Definition 2.8 *An operator L of $\mathcal{F}$ is called the zero operator of $\mathcal{F}$ if and only if $L \circ f = 0$ for all $f \in \mathcal{F}$.*

Definition 2.9 *The operators L_1 and L_2 of $\mathcal{F}$ are equal if and only if $L_1 - L_2$ is the zero operator of $\mathcal{F}$.*

Remark 2.7 Let $\mathcal{A}_p$ be the collection of all semi p-operators on $\mathcal{F}$. When the addition $+$ of semi p-operators is defined as in Definition 2.7, we have $p_1, p_2 \in \mathcal{A}_p \Rightarrow p_1 + p_2 \in \mathcal{A}_p$. In addition, we assume that the associative and the commutative law hold for the addition. It is clear that the zero operator defined in Definition 2.8 is a semi p-operator since it satisfies all the conditions in Definition 2.2. Since $\mathcal{F}$ is a linear space, $-p \circ f$ exists for any $f \in \mathcal{F}$ and any $p \in \mathcal{A}_p$ and $p_1 \circ f + p_2 \circ f = p_2 \circ f + p_1 \circ f$ holds for any $f \in \mathcal{F}$ and any $p_1, p_2 \in \mathcal{A}_p$. Now let us define the operator $-p$ such that $(-p) \circ f = -p \circ f \in \mathcal{F}$ for any $f \in \mathcal{F}$. Then $-p$ is also a semi p-operator since $-p$ satisfies all the conditions in Definition 2.2. Hence, $\mathcal{A}_p$ is an Abelian group.

Theorem 2.2 *Suppose that the topological space Y is a field. And let $\mathcal{A}$ be the collection of all functions $f : X \to Y$ such that addition and multiplication in $\mathcal{A}$ are induced from those in Y in a natural way:$(f+g)(x) \triangleq f(x) + g(x) = g(x) + f(x)$ and $(fg)(x) \triangleq f(x)g(x) = g(x)f(x)$. If $\mathcal{A}$ has some semi p-operator, then $\mathcal{A}$ is the function space $\mathcal{F}$ in Definition 2.1.*

Proof: Since Y is a field, $\mathcal{A}$ satisfies (**i**)-(**viii**) in Definition 2.1 by regarding that Y is also the coefficient field of $\mathcal{A}$. Hence, by Definition 2.1 and by assumption that some semi p-operator exists for $\mathcal{A}$, $\mathcal{A}$ is the function space $\mathcal{F}$ in Definition 2.1. Q.E.D.

Theorem 2.3 *Suppose that the topological space Y is a field. And let $\mathcal{F}$ be the collection of all bounded functions $f : X \to Y$ such that addition and multiplication in $\mathcal{F}$ are induced from those in Y in a natural way and $\mathcal{F}$ has the uniform norm $\|f\| = \sup_{x \in X} |f(x)|$. If Y is a Banach space and $\mathcal{F}$ has a continuous semi p-operator p, then $\mathcal{F}$ is a Banach space.*

Proof: By Theorem 2.2, $\mathcal{F}$ thus defined here is the function space $\mathcal{F}$ in Definition 2.1. Let $\{f_n\}$ be an arbitrary Cauchy sequence in $\mathcal{F}$. Namely for any given $\epsilon > 0$, there exists a positive integer n_0 such that $\|f_m - f_n\| < \epsilon$ for all $m, n \geq n_0$. Since $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|$, $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geq n_0$, where n_0 depends only on ϵ. If Y is complete with respect to its norm $|\cdot|$, any Cauchy sequence $\{f_m(x)\}$ in Y has the limit $f(x)$ in Y for any fixed $x \in X$ as $m \to \infty$. Hence, f is a mapping from X into Y. Then, by putting $m \to \infty$ in $|f_n(x) - f_m(x)| < \epsilon$, we have $|f_n(x) - f(x)| \leq \epsilon$ for all $n \geq n_0$, where n_0 depends only on ϵ. Since $|f_n(x) - f(x)| \leq \sup_{x \in X} |f_n(x) - f(x)| = \|f_n - f\| \leq \epsilon$, $\|f_n - f\| \to 0$ as $n \to \infty$.

Now let k be a non-negative integer and p be a continuous semi p-operator of $\mathcal{F}$. By Definition 2.2, if $f_n \in \mathcal{F}$, then $p^k \circ f_n \in \mathcal{F}$. And $p^k \circ f = \lim_{f_n \to f} p^k \circ f_n$ since p^k is a continuous operator. Therefore, if Y is complete, then $p^k \circ f : X \to Y$ exists. In addition, since p is a continuous operator, an operation of p to the case where the operand contains such limits can be defined by a usual limiting process of $f_n \to f$ such that **(i)** and **(ii)** in Definition 2.2 are still valid. Hence, $p^k \circ f \in \mathcal{F}$. Therefore, any Cauchy sequence $\{f_n\}$ in $\mathcal{F}$ has the limit f in $\mathcal{F}$ and Definition 2.2 still holds for this limit. Q.E.D.

Definition 2.10 *Let p be a semi p-operator of $\mathcal{F}$. $p^0 \circ f \triangleq 1 \circ f = f$ for any $f \in \mathcal{F}$, where 1 is the identity operator of $\mathcal{F}$.*

Remark 2.8 The identity operator 1 of $\mathcal{F}$ is different from the identity function $1 : X \to Y$ where the identity function maps all elements in X to the multiplicative identity element 1_Y in Y. By Definitions 2.6 and 2.10, we have $1 \circ L = L$ and $L \circ 1 = L$.

Lemma 2.2 *Let p be a semi p-operator of $\mathcal{F}$. Then, for any $f \in \mathcal{F}$ and $n = 0, 1, 2, \ldots$, we have $p^n \circ f \in \mathcal{F}$.*

Proof: By Definition 2.2, if $f \in \mathcal{F}$, then $p \circ f \in \mathcal{F}$. Hence, $f \in \mathcal{F} \Rightarrow p^2 \circ f \triangleq p \circ (p \circ f) \in \mathcal{F}$. Likewise, we can obtain $f \in \mathcal{F} \Rightarrow p^n \circ f \in \mathcal{F}$ for $n = 0, 1, 2, \ldots$, where $p^n \circ f \triangleq \underbrace{(p \circ (p \circ (\cdots (p}_{n \text{ times}} \circ f \underbrace{)\cdots)}_{n \text{ times}}$. Q.E.D.

Theorem 2.4 *Let p be a semi p-operator of $\mathcal{F}$. If f is such that $f \in \mathcal{F}$ and $\tilde{f}$ exists in $\mathcal{F}$, then λ_n which satisfies $p^n \circ f = \lambda_n f$ exists in $\mathcal{F}$ for $n = 0, 1, 2, \ldots$.*

Proof: Note that both $\tilde{f} \in \mathcal{F}$ and $p^n \circ f \in \mathcal{F}$ exist by the assumption and Lemma 2.2. Then, by (**ii**) in Definition 2.1, $\lambda_n \triangleq \tilde{f} p^n \circ f \in \mathcal{F}$ exists in $\mathcal{F}$ for $n = 0, 1, 2, \ldots$. Q.E.D.

Lemma 2.3 *Let p be a semi p-operator of $\mathcal{F}$ and let f be such that $f \in \mathcal{F}$ and $\tilde{f}$ exists in $\mathcal{F}$. And let λ_1 be such that $p \circ f = \lambda_1 f$. If $c \in \mathcal{F}$ is a non-zero constant function, then $f \in \mathcal{F}$ and $cf \in \mathcal{F}$ have the same λ_1.*

Proof: By (**i**) and (**ii**) in Definition 2.2,

$$p \circ (cf) = cp \circ f = c\lambda_1 f = \lambda_1 (cf).$$

Hence, $f \in \mathcal{F}$ and $cf \in \mathcal{F}$ have the same λ_1. Q.E.D.

Definition 2.11 *Let p be a p-operator of $\mathcal{F}$. And let $\mathcal{S}_p$ be $\mathcal{S}_p \triangleq \{g : g = p \circ f, f \in \mathcal{F}\}$. Then, Definition 2.2 states that for any $g \in \mathcal{S}_p$, there exists a unique $f \in \mathcal{F}$. For such an f, we write $f = p^{-1} \circ g$. We call p^{-1} "the inverse operator of p." Note that p^0 is itself its inverse.*

Theorem 2.5 *Let p be a p-operator of $\mathcal{F}$ and p^{-1} be the inverse operator of p. Then, $p^{-1} \circ p = 1$ for all $f \in \mathcal{F}$ and $p \circ p^{-1} = 1$ for all $g \in \mathcal{S}_p (\subset \mathcal{F})$. If p is linear, then p is an isomorphism between $\mathcal{F}$ and $\mathcal{S}_p$ under addition $+$.*

Proof: For any $f \in \mathcal{F}$, there exists a $g \in \mathcal{S}_p$ such that $f = p^{-1} \circ g$ and $p \circ f = g$. Then, we have $f = p^{-1} \circ g = p^{-1} \circ (p \circ f) = (p^{-1} \circ p) \circ f$ for any $f \in \mathcal{F}$. Hence, $p^{-1} \circ p = 1$ for all $f \in \mathcal{F}$.

Next, for any $g \in \mathcal{S}_p$, there exists a unique $f \in \mathcal{F}$ such that $p \circ f = g$ and $f = p^{-1} \circ g$. Therefore, $g = p \circ f = p \circ (p^{-1} \circ g) = (p \circ p^{-1}) \circ g$ for any $g \in \mathcal{S}_p$. Hence, $p \circ p^{-1} = 1$ for all $g \in \mathcal{S}_p$. In addition to $p^{-1} \circ p = p \circ p^{-1} = 1$, p is a homomorphism under addition $+$ if p is linear. Therefore, p is an isomorphism between $\mathcal{F}$ and $\mathcal{S}_p$ under addition $+$. Q.E.D.

Remark 2.9 If p is a linear semi p-operator, then for any $p \circ f, p \circ g \in \mathcal{S}_p$ and any scalars α,β, we have $\alpha p \circ f + \beta p \circ g = p \circ (\alpha f + \beta g)$. And we have $\alpha f + \beta g \in \mathcal{F}$ since $f, g \in \mathcal{F}$. So that $p \circ (\alpha f + \beta g) \in \mathcal{S}_p$. Hence $\mathcal{S}_p$ is a linear subspace of $\mathcal{F}$.

Remark 2.10 Let h be such that $h \in \mathcal{F}$ and $\tilde{h}$ exists in $\mathcal{F}$ and let p be a p-operator of $\mathcal{F}$. Then, it is easy to verify that $p^{-1}\tilde{h}$ is the inverse operator of hp.

Corollary 2.1 *Let p be a linear p-operator of $\mathcal{F}$ and let f and g be such that $f, g \in \mathcal{F}$. If either $g(p \circ f)$ or $f(p \circ g)$ belongs to $\mathcal{S}_p$, then*

$$fg = p^{-1} \circ \{g(p \circ f)\} + p^{-1} \circ \{f(p \circ g)\}. \tag{2.6}$$

Proof: Since $\mathcal{S}_p$ is a linear subspace of $\mathcal{F}$, $p \circ (fg), g(p \circ f), f(p \circ g) \in \mathcal{S}_p$ when either $g(p \circ f)$ or $f(p \circ g)$ belongs to $\mathcal{S}_p$. Then (2.6) is obvious from (**ii**) in Definition 2.2, Theorem 2.5, and the fact that p^{-1} is a linear operator to functions in $\mathcal{S}_p$ if p is a linear p-operator to functions in $\mathcal{F}$. Q.E.D.

Corollary 2.2 $\lambda_0 = \tilde{f} p^0 \circ f = 1$ *if $f \in \mathcal{F}$ and $\tilde{f}$ exists in $\mathcal{F}$.*

Proof: It is obvious from Definition 2.10 and (**viii**) in Definition 2.1. Q.E.D.

Theorem 2.6 *If $f \in \mathcal{F}$ and $\tilde{f}$ exists in $\mathcal{F}$, then λ_n is unique to f.*

Proof: Let us first set $p^n \circ f = \lambda_n f$ and $p^n \circ f = \lambda'_n f$ where p is a semi p-operator of $\mathcal{F}$. Then, we have

$$(\lambda_n - \lambda'_n) f = 0. \tag{2.7}$$

By applying $\tilde{f}$ to both sides of (2.7), we have

$$(\lambda_n - \lambda'_n) = 0.$$

Hence, λ_n is unique to f. Q.E.D.

Definition 2.12 *When $f, g \in \mathcal{F}$ are such that $\tilde{f}$ and $\tilde{g}$ exist in $\mathcal{F}$, we define $(\lambda_f)_n$ and $(\lambda_{fg})_n$ for $n = 0, 1, 2, \ldots$ by*

$$p^n \circ f = (\lambda_f)_n f$$

and

$$p^n \circ (fg) = (\lambda_{fg})_n fg,$$

respectively, where p is a semi p-operator of $\mathcal{F}$.

Theorem 2.7 *Let $f, g \in \mathcal{F}$ be such that $\tilde{f}$ and $\tilde{g}$ exist in $\mathcal{F}$. Then, $(\lambda_{fg})_1 = (\lambda_f)_1 + (\lambda_g)_1$.*

Proof: Let p be a semi p-operator of $\mathcal{F}$. By (**ii**) in Definition 2.2, we have

$$\begin{aligned} p \circ (fg) &= (p \circ f)g + f(p \circ g) \\ &= (\lambda_f)_1 fg + f(\lambda_g)_1 g \\ &= \{(\lambda_f)_1 + (\lambda_g)_1\} fg. \end{aligned}$$

Therefore, by Definition 2.12 and $fg\tilde{g}\tilde{f} = 1$, we have $(\lambda_{fg})_1 = (\lambda_f)_1 + (\lambda_g)_1$. Q.E.D.

Remark 2.11 Let f and g be such that $f, g \in \mathcal{F}$. If $\tilde{f}$ and $\tilde{g}$ exist for f and g, respectively, then $p \circ f = 0 \Leftrightarrow (\lambda_f)_1 = 0$ and $p \circ (fg) = (p \circ f)g + f(p \circ g) \Leftrightarrow (\lambda_{fg})_1 = (\lambda_f)_1 + (\lambda_g)_1$. By the former, we can say that if f is a non-zero constant function, then $(\lambda_f)_1 = 0$. This statement corresponds to (i) in Definition 2.2. By the latter, $(\lambda_{fg})_1 = (\lambda_f)_1 + (\lambda_g)_1$ corresponds to (ii) in Definition 2.2.

Theorem 2.8 (Leibniz's Theorem) *For $f, g \in \mathcal{F}$ such that $\tilde{f}$ and $\tilde{g}$ exist in $\mathcal{F}$ and for a linear semi p-operator p of $\mathcal{F}$,*

$$p^n \circ (fg) = \{\sum_{k=0}^{n} \binom{n}{k} (\lambda_f)_k (\lambda_g)_{n-k}\} fg, \tag{2.8}$$

where $n = 0, 1, 2, \ldots$, $\binom{n}{k}$ is

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{2.9}$$

for $k = 1, 2, \ldots, n$, and

$$\binom{n}{0} = 1. \tag{2.10}$$

Proof: First, it is clear that (2.8) is valid for $n = 0, 1$ by Definition 2.10, Corollary 2.2, and Theorem 2.7. Assume that (2.8) holds for $n = k$. Then, since p is linear, we have

$$p \circ \{p^k \circ (fg)\} \quad = \quad p \circ \left[\left\{ \sum_{j=0}^{k} \binom{k}{j} (\lambda_f)_j (\lambda_g)_{k-j} \right\} fg \right]$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \binom{k}{j} p \circ [\{(\lambda_f)_j f\}\{(\lambda_g)_{k-j} g\}] \\
&= \sum_{j=0}^{k} \binom{k}{j} (\lambda_g)_{k-j} g \cdot p \circ \{(\lambda_f)_j f\} \\
&\qquad + \sum_{j=0}^{k} \binom{k}{j} (\lambda_f)_j f \cdot p \circ \{(\lambda_g)_{k-j} g\} \\
&= \sum_{j=1}^{k+1} \binom{k}{j-1} (\lambda_g)_{(k+1)-j} g \cdot (\lambda_f)_j f \\
&\qquad + \sum_{j=0}^{k} \binom{k}{j} (\lambda_f)_j f \cdot (\lambda_g)_{(k+1)-j} g \\
&= (\lambda_f)_0 (\lambda_g)_{k+1} fg + (\lambda_g)_0 (\lambda_f)_{k+1} fg \\
&\quad + \sum_{j=1}^{k} \left\{ \binom{k}{j} + \binom{k}{j-1} \right\} (\lambda_f)_j (\lambda_g)_{(k+1)-j} fg \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} (\lambda_f)_j (\lambda_g)_{(k+1)-j} fg. \qquad (2.11)
\end{aligned}
$$

By mathematical induction, we proved that (2.8) holds for $n = 0, 1, 2, \ldots$. Q.E.D.

Remark 2.12 (2.8) in Theorem 2.8 means

$$
\frac{(\lambda_{fg})_n}{\Gamma(n+1)} = \sum_{k=0}^{n} \frac{(\lambda_f)_k}{\Gamma(k+1)} \frac{(\lambda_g)_{n-k}}{\Gamma(n-k+1)} \qquad (2.12)
$$

which can also be written as

$$
\frac{(\lambda_{fg})_n}{\Gamma(n+1)} = \frac{(\lambda_f)_n}{\Gamma(n+1)} * \frac{(\lambda_g)_n}{\Gamma(n+1)}, \qquad (2.13)
$$

where $*$ denotes the discrete convolution and $\Gamma(\cdot)$ is the Gamma function.

Remark 2.13 [22] Since

$$
\frac{1}{\Gamma(k+1)} = 0
$$