A UNIFIED THEORY OF GENERALIZED DIFFERENTIATION AND INTEGRATION

—A NEW APPROACH TO FRACTIONAL CALCULUS—

Third Edition

Takahiro INOUE

A UNIFIED THEORY OF GENERALIZED DIFFERENTIATION AND INTEGRATION

—A NEW APPROACH TO FRACTIONAL CALCULUS—

Third Edition

Takahiro INOUE

Professor Emeritus, Kumamoto University

Dedicated to my wife Toshiko.

Preface to the First Edition

The aim of this book is to give a unified theory of generalized differentiation and integration. In this book, a differentiator-like operator to arbitrary order has been defined and a theory on fractional calculus has been developed from a new standpoint. It is well-known that classical differentiation and integration to arbitrary order have been well established in classical fractional calculus[1]-[3] and in distribution theory[21],[22]. The theory presented in this book is constructed on a more general operator-based formulation which includes not only the previous formulations of fractional calculus but also a new formulation based on general differentiator-like linear operators to arbitrary order.

The incentive to develop this theory dates back to my master thesis. When I was a senior student in the undergraduate course of the Department of Electrical Engineering, Kumamoto University(KU), I began my theoretical research on a wave propagation problem so-called the Sommerfeld Problem under guidance of Prof. A. Yokoyama. At that time, Prof. Yokoyama developed a very efficient operational method to derive an asymptotic formula for a far-zone field of a dipole radiation effected by the plane earth. I was very much impressed by the beauty of his method and I tried to found his method on a more rigorous mathematical base in my master thesis. My main idea is to consider a sequence $\{\lambda_n\}_{n=0}^{\infty}$ generated by the equation $p^n \circ f = \lambda_n f$, where f and λ_n are functions and p is a differentiator-like operator. In my master thesis, I derived the main theorem in Chapter 3 in this book and applied it to the Sommerfeld Problem. When I was preparing my master thesis, I noticed that, since the Gamma function can be regarded as interpolation of a factorial, it would be possible to extend differentiation and integration to arbitrary order by using the Gamma functions. So I dreamed to extend λ_n to non-integer index values in order to make it easier to evaluate contour integrals around a pole or a branch point that we encounter in the Sommerfeld Problem. After graduating from the Graduate School of Electrical Engineering in KU, I worked for Hitachi Ltd. in the field of an IC technology as a research staff. About four years later, I joined the Faculty of Engineering, KU, where I engaged in education and research on electronic circuits until my retirement. During these periods, I had been almost apart from my dream though I sometimes continued my private research on weekends. After my retirement from KU, I started again on a theoretical work to pursue my dream. At the end of 2014, I decided to write a book about my works on constructing a unified theory of generalized differentiation and integration by extending $p^n \circ f = \lambda_n f$ to arbitrary order. To tell the truth, I have been in complete ignorance of the fact that fractional calculus has already been well developed and it has a three-century history. I learned fractional calculus from a book written by Oldham and Spanier[1] recently. From January to May, 2015, I was so busy and nervous because I had to check whether the results given in Oldham & Spanier's book can be reproduced from my theory. Finishing this work, I convinced that my theory can reproduce the classical results and it is an extension of classical fractional calculus.

In this book, a differentiation operator in fractional calculus is generalized to a more general operator that I call semi p-operator which may be linear or nonlinear. Then, we consider a function sequence $\{\lambda_n\}_{n=0}^{\infty}$ generated by the equation $p^n \circ f = \lambda_n f$ and introduce its integral representation in a complex region. Like the Gamma function which is an extension of a factorial, we can extend λ_n to arbitrary index values via its integral representation. Thereby we can define a linear semi p-operator to arbitrary order. The classical fractional calculus can be derived as a special case and various special functions of arbitrary order follow from λ_z by choosing different pairs of p and f. Some applications of the theory are given in the last chapter with special emphasis on the electromagnetic wave problems.

Chapter 1 in this book is a brief introduction of the history of fractional calculus. The following Chapter 2 is the preliminaries which give definitions of a function space and operators to be used in the succeeding chapters. In Chapter 3, the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is introduced and the fundamental theorems are proved. In Chapter 4, more general function sequences $\{\mu_n\}_{n=0}^{\infty}$'s and the sequences induced by them are considered and discussed in three interesting cases. It is proved that those induced sequences have expansions of a similar form as λ_n has. The integral representation of λ_n and its extension to arbitrary order are described in Chapter 5. In this chapter, the relations to the previous definitions of fractional calculus are discussed. In Chapter 6, various special functions are derived as λ_n for different pairs of p and f, and they are extended to arbitrary order. Some applications of the theory are given in Chapter 7 where an order-reduction of ordinary differential equations of a certain type, diffusion equations, a semi-infinite electric circuit, and electromagnetic wave problems are investigated. The mathematical proofs for the procedure in deriving an approximate solution for the Sommerfeld Problem are given in Appendix A.

Finally, I want to express my sincere gratitude to Prof. Akira Yokoyama for guiding me to this fruitful field of science and I must also express my special thanks to my wife Toshiko for her continuous encouragement to this work.

Takahiro Inoue

Kumamoto
October, 2015

Preface to the Second Edition

In this second edition, I corrected the errata in the first edition throughout this book and added some new theorems and remarks to clarify the meanings and to present further properties of differintegration. In particular, Definition 2.1 is refined after clarifying two operations which are implicitly assumed for the topological space Y. Theorem 5.7 is added to present the relation that holds between the generating function of λ_n and its partial sum. Theorem 5.12 is added to show the shift-invariance property of the operator $(d/dz)^\alpha$ where α is a complex order. Case I is added in Section 5.5 to show that the extension of $(\gamma D)^n \circ f = \lambda_n f$ to arbitrary order can be derived from its induced sequence. Theorem 6.5 is added to show that the composition rule holds for the operator $(d/dz)^\alpha$ when it is applied to complex functions which are regular in an open disk around a fixed point in the complex plane. Theorem 7.1 is added to show a general solution of a basic differintegral equation. Subsection 7.3.3 is added to show the derivation of the quasi-classical approximation(the WKB approximation) for a solution of Schrödinger's equation in quantum mechanics. The improved Definitions, Theorems, Corollaries, and Remarks are Theorems 2.2-2.6, Remarks 2.6-2.7, Remark 2.9, Corollary 3.2, Remarks 3.3-3.4, Theorem 3.4, Sections 4.1-4.3, Definition 5.1, Remark 5.2, Remarks 5.4-5.5, Remark 5.7, Theorem 5.13, Theorem 5.18, Corollary 5.12, Remark 5.13, Subsections 5.5.2-5.5.3, Definition 6.1, Remarks 6.1-6.5, and Subsection 6.3.1.

Finally, I want to express my thanks to my wife Toshiko for her continued support to this work.

Takahiro Inoue

Kumamoto
May, 2016

Preface to the Third Edition

In this third edition, not a few parts of the definitions and proofs in the previous editions are improved to clarify their concepts and logic, and the errata which still remained in the second edition are corrected. Especially, the erratum in Theorem 5.2 is corrected and the related Remark 5.4 is revised. Definition 5.2 is revised in regard to the condition for the operator $(\partial/\partial\lambda_1)^\alpha$ (where α is a complex order) to be a homogeneous operator. Regarding the sufficient condition for the operator $(d/dx)^\alpha$ to be a continuous operator, the erratum in Theorem 5.14 is corrected. And by Theorem 5.16, a weaker sufficient condition is given for $(d/dx)^\alpha$ to be a continuous operator to bounded regular functions. In Theorem 6.3 and Corollaries 6.4-6.5, the conditions for the exception are revised. And in regard to the problem of $(d/dx)^{-1} \circ x^{-1}$ discussed at the end of the proof of Theorem 6.3, an incorrect remedy suggested for this problem in the previous editions is excised. Theorem 7.1 is revised to exclude ambiguity and it is supplemented by Remarks 7.1-7.2. Theorems A.1 and A.2 in Appendix A are also revised to clarify their respective proofs.

The materials newly added in this edition are Definition 2.4, Remark 2.5, Theorem 2.1, Remark 2.8, Remark 2.14, Theorem 3.10, Remark 5.11, Theorem 5.16, Remark 5.12, and Remarks 7.1-7.2. Theorem 2.1 shows that the factor group $\mathcal{F}_\infty/\mathcal{N}_\infty$ can be identified with the space $\mathcal{F}_\infty$ under addition and convolution, where $\mathcal{F}_\infty$ is the extension of the space $\mathcal{F}$ generated by adding the delta function δ and its higher derivatives $p^n \circ \delta$ to $\mathcal{F}$ and $\mathcal{N}_\infty$ is the normal subgroup given by $\mathcal{N}_\infty = \{f : f \in \mathcal{F}_\infty, p \circ f = 0\}$. Remark 2.14 shows that we can obtain weighted Leibniz's rule by extending original Leibniz's rule when the weight is a constant and the multiplication operation is the convolution. Theorem 3.10 gives the derivation of λ_m for when a semi p-operator is a sum of n different semi p-operators and it is applied to a product of n different functions in $\mathcal{F}$. Remark 5.11 shows that Riemann's zeta function $\varsigma(\beta + 1)$ can be obtained by the β-fold fractional integration of $x/(e^x - 1)$ for when $x = 0$.

Finally, I want to express my thanks to my wife Toshiko for her encouragement to this work.

Takahiro Inoue

Kumamoto
October, 2016

Contents

Chapter 1

Introduction

Interpolation of differentiation and integration to non-integer order is often called "*fractional differintegration*" and a calculus based on this is called "*fractional calculus*" [1]-[3]. Since this interpolation can be extended to arbitrary complex order by analytic continuation, the theory on fractional calculus is the generalization and unification of differentiation and integration.

The history of fractional calculus can be traced back to Leibniz's response to a letter of L'Hospital's dated 30th September, 1695[4]. L'Hospital asked him what would result if $n = 1/2$ for $d^n f(x)/dx^n$. Leibniz responded prophetically that one day useful consequences will be drawn from it[1],[2]. In 1819, S. F. Lacroix[5] obtained

$$(\frac{d}{dx})^{1/2} x = \frac{2\sqrt{x}}{\sqrt{\pi}} \tag{1.1}$$

by extending integers m and n of

$$(\frac{d}{dx})^n x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-n)} \cdot x^{m-n} \tag{1.2}$$

to $m = 1$ and $n = 1/2$, where $\Gamma(\cdot)$ is the Gamma function. In 1822, J. B. J. Fourier[6] generalized differentiation and integration by regarding n of the formula

$$(\frac{d}{dx})^n f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} \beta^n \cos(\beta x - \beta\alpha + \frac{n\pi}{2}) d\beta \tag{1.3}$$

as any real number. In 1823, N. H. Abel[7],[8] solved the tautochrone problem by using fractional integration of order 1/2. From 1832 to 1855, J. Liouville[9]-[11] carried out major studies on fractional calculus. He expanded functions in series of exponential functions and defined the non-integer order derivatives by term-by-term fractional operations. He also discussed fractional derivatives as the limits of a discretized-version definition of fractional derivatives. In 1847, B. Riemann[12] derived the definition of fractional integration

$$(\frac{d}{dx})^{-q} f(x) = \frac{1}{\Gamma(q)} \int_{c}^{x} (x-y)^{q-1} f(y) dy, \tag{1.4}$$

where $q > 0$. In 1884, H. Laurent[13] generalized Cauchy's integral formula (or Goursat's formula) for the n-th derivative in a complex variable by extending Letnikov's work in

1872[14]. The unified discretized-version of fractional differentiation and integration was given by A. K. Grünwald in 1867[15] and extended by E. L. Post in 1930[16]. This unified discretized-version of fractional differentiation and integration is proven to be equivalent to the Riemann-Liouville definition:

$$(\frac{d}{dx})^q f(x) = \frac{1}{\Gamma(n-q)}(\frac{d}{dx})^n \int_a^x (x-y)^{-q+n-1} f(y)dy, \tag{1.5}$$

where n is an integer such that $(n-1) \leq q < n$[2]. In 1899, O. Heaviside used fractional differentiation in his transmission line theory[17]. Though the rigorousness of the operational calculus developed by him had been criticized, it was later founded on a rigorous mathematical ground by J. Mikusiński[18]. Readers will find further detailed history and information on recent developments in References [1]-[3].

Fractional calculus provides us with a very powerful mathematical tool in analyzing natural phenomena in the real world since many natural phenomena we encounter in the real world are governed by the power law. The usefulness of fractional calculus is in that the Fourier transform of a power law distribution in the time domain (e.g., a long-tailed decay) is also a power law in the frequency domain. Thereby the fractional differintegration by time is directly related to the fractional power law in the frequency domain by the inverse Fourier transform. An interesting discussion on this matter can be found in Reference [2]. Another important fact regarding fractional calculus is that differentiation to non-integer order is based on a nonlocal property of a function to be differentiated whereas differentiation to integer order is based on a local property of the function. This is because fractional differentiation to non-integer order is essentially the interpolation given by an integral representation which depends on a nonlocal property of the integrand.

In the following chapters of this book, a differential operator is extended to a more general differentiator-like operator which I call "*semi p-operator*" in this book. In case where this semi p-operator is linear, the integer-order operations of this operator are extended to arbitrary order for functions belonging to a sufficiently broad function space. In this theory, the sequence which I call the "*λ-sequence*" in this book and its contour integral representation are essential. By using these concepts, the relation between the proposed definition and the classical definitions of fractional differintegrations is explained. In later chapters, applications to special functions and to some typical problems in physics are also presented.

Chapter 2

Preliminaries

2.1 Function Space $\mathcal{F}$ and Related Operators

Let X and Y be topological spaces. In addition, we assume that Y has the zero element 0_Y and the multiplicative identity element 1_Y under addition and multiplication in Y, respectively.
(Throughout this book, 0_Y and 1_Y will be simply denoted by 0 and 1, respectively, wherever they will not be confused with the zero function 0 and the identity function 1, both of which are the elements of the following function space $\mathcal{F}$.)

Definition 2.1 *A "space $\mathcal{F}$" is a collection of functions $f : X \to Y$ such that*
(**i**) *$\mathcal{F}$ is a linear space;*
(**ii**) *$\mathcal{F}$ is closed under multiplication:$fg \in \mathcal{F}$ for any $f, g \in \mathcal{F}$;*
(**iii**) *Multiplication is commutative:$fg = gf$ for any $f, g \in \mathcal{F}$;*
(**iv**) *Multiplication is associative:$(fg)h = f(gh)$ for any $f, g, h \in \mathcal{F}$;*
(**v**) *Scalar multiplication is defined such that $(\alpha f)(\beta g) = (\alpha\beta)(fg) \in \mathcal{F}$ for any scalars α and β;*
(**vi**) *Addition and multiplication in $\mathcal{F}$ are connected by the distributive law: $(f + g)h = fh + gh$ for any $f, g, h \in \mathcal{F}$;*
(**vii**) *$\mathcal{F}$ has the identity function 1 such that $1f = f1 = f$ for any $f \in \mathcal{F}$;*
(**viii**) *For any $f \in \mathcal{F}$ such that $0_Y \notin f(X)$, there exists a unique $\tilde{f} \in \mathcal{F}$ such that $\tilde{f}f = f\tilde{f} = 1$.*

In addition, we assume $\mathcal{F}$ has at least one (semi) p-operator defined below.

Definition 2.2 *A "semi p-operator" p is a mapping of $\mathcal{F}$ into itself such that*
(**i**) *$p \circ c = 0$ if and only if $c(\in \mathcal{F})$ is a constant function;*
(**ii**) *$p \circ (fg) = (p \circ f)g + f(p \circ g)$ for any $f, g \in \mathcal{F}$.*
In case where p can be defined such that, under a certain definite rule, a unique $f \in \mathcal{F}$ is specified corresponding to each $g(= p \circ f) \in \mathcal{F}$, I call it a "p-operator" of $\mathcal{F}$.
(Note here that $p \circ 1 = 0$ can be derived only from (**ii**) of Definition 2.2 if some $f \in \mathcal{F}$ has $\tilde{f}$. Hence, in Definition 2.2, (**i**) can be replaced with $[p, c] = 0$ if some $f \in \mathcal{F}$ has $\tilde{f}$ where $[\cdot, \cdot]$ is the commutator defined in Sec. 2.2.)

Definition 2.3 *If $p \circ f_n \to p \circ f$ as $f_n \to f$ for any $f_n, f \in S(\subset \mathcal{F})$, then p is called a continuous (semi) p-operator on S.*

Remark 2.1 Let us define df for $f \in \mathcal{F}$ by $df \triangleq Hf - fH$, where H is an operator such that $H : \mathcal{F} \to \mathcal{F}$ and $Hc = cH$ for any constant c. Then, the operator d is a semi p-operator.

Remark 2.2 An example of a semi p-operator in tensor analysis is the covariant derivative under which the fundamental tensor and the unit tensor are constant functions.

Remark 2.3 Let h be such that $h \in \mathcal{F}$ and let p be a semi p-operator of $\mathcal{F}$. Then, it is easy to verify that hp is also a semi p-operator of $\mathcal{F}$.

Remark 2.4 Note that when p is a *linear* semi p-operator, $p \circ (f + c) = p \circ f$ holds for any $f \in \mathcal{F}$ and any constant function c. Then, Definition 2.2 means that if we specify f such that $f(0) = 0$, then f corresponding to a given $g(= p \circ f)$ becomes unique. Then, p becomes a p-operator. From a different viewpoint, let us construct an equivalent relation $\sim$ in X[19] such that for $f, h \in \mathcal{F}$, $f \sim h$ if and only if $p \circ f = p \circ h$. And let $\mathbf{F}$ denote the class of all distinct equivalence sets[19] generated by $\sim$. Then, it is clear that there exists a unique equivalence set in $\mathbf{F}$ to a given $g(= p \circ f)$. Such an equivalent set unique to f is an inverse image $p^{-1}(\{p \circ f\})(\triangleq \{g : g \in \mathcal{F}, p \circ g = p \circ f\})$. $\mathbf{F}$ is also the factor group $\mathcal{F}/\mathcal{N}$[19] under addition of $\mathcal{F}$, where $\mathcal{N}$ is the normal subgroup given by $\mathcal{N} = \{f : f \in \mathcal{F}, p \circ f = 0\}$. (For the definition of the *linearity* of operators, refer to (**iv**) in Definition 2.5 below.)

Definition 2.4 *Suppose that p is a linear semi p-operator. Then, let $\mathcal{F}_\infty$ be the extension of $\mathcal{F}$ generated by adding $\delta, p \circ \delta, p^2 \circ \delta, \ldots$ to $\mathcal{F}$, where δ is the delta function defined such that $\delta \odot f = f$ for any $f \in \mathcal{F}_\infty$, $\tilde{\delta} = \delta$, and $\delta, p \circ \delta, p^2 \circ \delta, \ldots \notin \mathcal{N}_\infty$. Here, $\mathcal{N}_\infty$ denotes $\mathcal{N}_\infty \triangleq \{f : f \in \mathcal{F}_\infty, p \circ f = 0\}$. A binomial operation $\odot : \mathcal{F}_\infty \times \mathcal{F}_\infty \to \mathcal{F}_\infty$ is defined such that $p \circ (f \odot g) = \rho\{(p \circ f) \odot g + f \odot (p \circ g)\}$ for any $f, g \in \mathcal{F}_\infty$ (weighted Leibniz's rule), where ρ denotes the weight, and $c \odot f \in \mathcal{N}_\infty$ for any $c \in \mathcal{N}_\infty$ and $f \in \mathcal{F}_\infty$. For $f \notin \mathcal{N}_\infty$, $\tilde{f} \in \mathcal{F}_\infty$ is defined such that $\tilde{f} \odot f = f \odot \tilde{f} = \delta$. And we further assume that $\odot$ satisfies the commutative, the associative, and the distributive law. Note here that since $\delta, p \circ \delta, p^2 \circ \delta, \ldots$ are generalized functions, $\mathcal{F}_\infty$ must be considered as a collection of generalized functions*[21], [22] *whereas $\mathcal{F}$ is a collection of functions. We further define $\mathbf{F}_\infty$ by $\mathbf{F}_\infty \triangleq \mathcal{F}_\infty/\mathcal{N}_\infty$.*

Remark 2.5 Let $\mathcal{F}_\infty$ be the collection of all real generalized functions f's. And let a binomial operation $\odot$ be the convolution $*$ of $f, g \in \mathcal{F}_\infty$;

$$f * g \triangleq \int_{-\infty}^{+\infty} f(\frac{x}{2} - \varsigma) g(\frac{x}{2} + \varsigma) d\varsigma.$$

Then, $p^n \circ f$ can be defined by $p^n \circ f = \delta^{(n)} * f = f^{(n)}$[20], where $f^{(n)}$ denotes the n-th derivative of $f \in \mathcal{F}_\infty$. Under this definition, it is clear that $\delta^{(m)} * \delta^{(n)} = \delta^{(m+n)}$ for all nonnegative integers m, n. And it is easy to see that $f * g$ satisfies weighted Leibniz's rule with $\rho = 1/2$:

$$p \circ (f * g) = \frac{1}{2}\{(p \circ f) * g + f * (p \circ g)\}.$$

Under all these definitions, it is clear that such an $\mathcal{F}_\infty$ satisfies all the requirements mentioned in Definition 2.4

Theorem 2.1 *If for each $F(\neq \mathcal{N}_\infty) \in \mathbf{F}_\infty$, there exists some $h \in F$ such that $x * h = \delta$ is solvable in $\mathcal{F}_\infty$, then $\mathbf{F}_\infty$ in Definition 2.4 can be identified with $\mathcal{F}_\infty$ under addition $+$ and multiplication $*$(the convolution). And a linear p-operator which maps $\mathbf{F}_\infty$ one-to-one onto itself can be induced from a linear semi p-operator of $\mathcal{F}_\infty$ provided that weighted Leibniz's rule is adopted instead of* (**ii**) *in Definition 2.2.*

Proof: For any $F_1, F_2 \in \mathbf{F}_\infty$, let us define addition $+$ by

$$F_1 + F_2 = \{g : p \circ g = p \circ (f_1 + f_2); g \in \mathcal{F}_\infty, f_1 \in F_1, f_2 \in F_2\} \equiv \{f_1 + f_2 + c_1\}, \quad (2.1)$$

where c_1 is an arbitrary element of $\mathcal{N}_\infty$. And for multiplication $*$(the convolution) in $\mathcal{F}_\infty$, we define it by

$$F_1 * F_2 = \{g : p \circ g = p \circ (f_1 * f_2); g \in \mathcal{F}_\infty, f_1 \in F_1, f_2 \in F_2\} \equiv \{f_1 * f_2 + c_2\}, \quad (2.2)$$

where c_2 is an arbitrary element of $\mathcal{N}_\infty$. In addition, let us define E by

$$E = \{g : p \circ g = p \circ \delta; g \in \mathcal{F}_\infty\} \equiv \{\delta + c_3\}, \quad (2.3)$$

where c_3 is an arbitrary element of $\mathcal{N}_\infty$. And for $F \neq \mathcal{N}_\infty$, we define $\tilde{F}$ by

$$\begin{aligned} \tilde{F} &= \{g : p \circ g = p \circ x; g, x \in \mathcal{F}_\infty, x * h = \delta \text{ for some } h \in F\} \\ &\equiv \{x + c_4 : x * h = \delta \text{ for some } h \in F\}, \end{aligned} \quad (2.4)$$

where c_4 is an arbitrary element of $\mathcal{N}_\infty$. Since $\tilde{\delta} = \delta$, it is clear that $\tilde{E} = E$.
(Note that by Titchmarch's theorem[18] and $h \neq 0$, $x \in \mathcal{F}_\infty$ is unique to $h \in F$ which satisfies $x * h = \delta$. We write such an x as h^{*-1}. If $x * h_1 = \delta$ and $x' * (h_1 + c_1) = \delta$ for $h_1 \in F$ and $c_1 \in \mathcal{N}_\infty$, then $x' = x + c$ where $c = -c_1 * (h_1 + c_1)^{*-1} * h_1^{*-1} \in \mathcal{N}_\infty$. Hence, $x \in \tilde{F} \Rightarrow x' \in \tilde{F}$.)
By Definitions 2.1 and 2.4, It is clear that addition $+$ and multiplication $*$ of F_1 and F_2 satisfy the commutative, the associative, and the distributive law. And it is also clear that $\mathcal{N}_\infty \in \mathbf{F}_\infty$ is the zero element under addition $+$. By Definition 2.4 and Remark 2.5 and from (2.2) and (2.3), we have for $E, F \in \mathbf{F}_\infty$ and $f \in F$,

$$\begin{aligned} E * F &= \{(\delta + c_1) * (f + c_2) + c_3\} \\ &= \{\delta * f + c_1 * f + c_2 * \delta + c_1 * c_2 + c_3\} \\ &= \{\delta * f + c\} \\ &= \{f + c\} \\ &= F, \end{aligned} \quad (2.5)$$

where c_1, c_2, c_3, and c are arbitrary elements of $\mathcal{N}_\infty$. Now that $E * F = F * E = F$, E is the identity element of $\mathbf{F}_\infty$. By Definition 2.4 and Remark 2.5 and from (2.2) and (2.4), if $F \neq \mathcal{N}_\infty$ and $f \in F$, then

$$\begin{aligned} \tilde{F} * F &= \{(x + c_1) * (f + c_2) + c_3\} \\ &= \{x * f + c_1 * f + c_2 * x + c_1 * c_2 + c_3\} \\ &= \{x * f + c\} \\ &= \{x * (h + c') + c\} \\ &= \{x * h + c' * x + c\} \\ &= \{x * h + c"\} \\ &= \{\delta + c"\} \\ &= E, \end{aligned} \tag{2.6}$$

where c_1, c_2, c_3, c, and $c"$ are arbitrary elements of $\mathcal{N}_\infty$ and $c' \in \mathcal{N}_\infty$ is such that $f = h + c'$ for $h \in F$ in (2.4). Hence, $\tilde{F} \in \mathbf{F}_\infty$ is the counterpart of $\tilde{f} \equiv f^{*-1} \in \mathcal{F}_\infty$ satisfying $\tilde{f} * f = \delta$. From above discussions, $\mathbf{F}_\infty$ can be identified with $\mathcal{F}_\infty$ except for the operator p. Next, let us define a function $\phi : \mathcal{F}_\infty \to \mathbf{F}_\infty$ by

$$\phi(f) = F \text{ if and only if } f \in F, \tag{2.7}$$

where $f \in \mathcal{F}_\infty$ and $F \in \mathbf{F}_\infty$. Since $\mathbf{F}_\infty$ is a partition of $\mathcal{F}_\infty$, for $f_1 \in F_1 (\in \mathbf{F}_\infty)$ and $f_2 \in F_2 (\in \mathbf{F}_\infty)$ we have

$$\phi(\alpha f_1 + \beta f_2) = \alpha F_1 + \beta F_2 = \alpha \phi(f_1) + \beta \phi(f_2), \tag{2.8}$$

where α and β are arbitrary scalars. Hence, ϕ is a linear function. Now, let us define $\hat{p} \circ F$ such that

$$\hat{p} \circ F = \phi(p \circ f) \tag{2.9}$$

if and only if $f \in F (\in \mathbf{F}_\infty)$ and $p \circ f \in \hat{p} \circ F (\in \mathbf{F}_\infty)$. It is clear that $\phi \circ p$ is linear since both p and ϕ are linear. Hence, $\hat{p}$ is a linear operator which maps $\mathbf{F}_\infty$ into itself.

Suppose that for $f_1 \in F_1$ and $f_2 \in F_2$, $p \circ f_2 = p \circ f_1$. Then, $f_2 \in F_1$. Therefore, $F_1 = F_2$. Hence $F_1 \neq F_2 \Rightarrow p \circ f_1 \neq p \circ f_2$. Next, if $\hat{p} \circ F_1 = \hat{p} \circ F_2$, then we can choose the same element of $\hat{p} \circ F_1 = \hat{p} \circ F_2$ such that $p \circ f_2 = p \circ f_1$. Therefore, $F_1 = F_2$. Hence, $\hat{p} \circ F_1 = \hat{p} \circ F_2 \Rightarrow F_1 = F_2$. Consequently, $\hat{p}$ is a linear one-to-one operator.

Lastly, we prove that weighted Leibniz's rule holds for $\hat{p}$. By (2.8), (2.9), and Remark 2.5, we have for $f_1 \in F_1 (\in \mathbf{F}_\infty)$ and $f_2 \in F_2 (\in \mathbf{F}_\infty)$

$$\begin{aligned} \hat{p} \circ (F_1 * F_2) &= \phi(p \circ (f_1 * f_2)) \\ &= \phi(\frac{1}{2}(p \circ f_1) * f_2 + \frac{1}{2} f_1 * (p \circ f_2)) \\ &= \frac{1}{2}\phi((p \circ f_1) * f_2) + \frac{1}{2}\phi(f_1 * (p \circ f_2)). \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} (\hat{p} \circ F_1) * F_2 &= \{(p \circ f_1) * f_2 + c_1\} = \phi((p \circ f_1) * f_2) \\ F_1 * (\hat{p} \circ F_2) &= \{f_1 * (p \circ f_2) + c_2\} = \phi(f_1 * (p \circ f_2)), \end{aligned} \tag{2.11}$$

where $f_1 \in F_1(\in \mathbf{F}_\infty)$ and $f_2 \in F_2(\in \mathbf{F}_\infty)$ and c_1, c_2 are arbitrary elements of $\mathcal{N}_\infty$, we have

$$\hat{p} \circ (F_1 * F_2) = \frac{1}{2}\{(\hat{p} \circ F_1) * F_2 + F_1 * (\hat{p} \circ F_2)\}. \tag{2.12}$$

Q.E.D.

Theorem 2.2 *Let $\mathcal{F}$ be the set of all infinitely smooth real functions $f : X \to Y$ where both X and Y are in $(-\infty, +\infty)$. And let p be a semi p-operator of $\mathcal{F}$. Then, for all $x \in X$ such that $f(x) \neq 0$, $p \circ f$ can be expressed as*

$$p \circ f = (fL) \circ (\frac{d}{dx}) \circ \ln |f|, \tag{2.13}$$

where L is a linear operator of $\mathcal{F}$ (see Definition 2.5).

Proof: By (**ii**) in Definition 2.2, we have

$$\frac{p \circ (fg)}{fg} = \frac{p \circ f}{f} + \frac{p \circ g}{g} \tag{2.14}$$

for all $x \in X$ such that $f(x) \neq 0$ and $g(x) \neq 0$ where $f, g \in \mathcal{F}$. Since (2.14) implies the logarithmic operation for the product of functions, there exists some linear operator $L_0 : \mathcal{F} \to \mathcal{F}$ such that

$$\frac{p \circ f}{f} = L_0 \circ \ln |f|. \tag{2.15}$$

In addition, to satisfy (**i**) in Definition 2.2, the linear operator L_0 can be written as

$$L_0 = L \circ (\frac{d}{dx}), \tag{2.16}$$

where L is a linear operator of $\mathcal{F}$. Q.E.D.

Remark 2.6 Let $\mathcal{F}$ be the set of all regular functions $f(z)$'s defined on a complex region $\mathcal{D}$. And let p be a semi p-operator of $\mathcal{F}$. Then, for all $z \in \mathcal{D}$ such that $f(z) \neq 0$, $p \circ f$ can be expressed as

$$p \circ f = (fL) \circ (\frac{d}{dz}) \circ \ln f, \tag{2.17}$$

where L is a linear operator of $\mathcal{F}$.
Note here that for $f : X \to Y$, $X = D$ and Y is the complex plane.

Remark 2.7 Let $\mathcal{F}$ be the set of all infinitely smooth real functions $f : X \to Y$ where both X and Y are in $(-\infty, +\infty)$. For $f \in \mathcal{S} \equiv \{f : f \in \mathcal{F}, f(x) \neq 0 \text{ for all } x \in X\}$, let us define s_n by

$$s_n \circ f = f(x)(d/dx)^n \ln |f(x)|,$$

where n is a positive integer. Then, s_n is an operator $s_n : \mathcal{S} \to \mathcal{F}$. Note that if $n = 1$, then the operator s_1 can be extended to a linear semi p-operator of $\mathcal{F}$ since $f(x)(d/dx) \ln |f(x)| = (d/dx)f(x)$. If $n = 2$, then s_2 is a nonlinear operator since

$$f(x)(d/dx)^2 \ln |f(x)| = (d/dx)^2 f(x) - \{1/f(x)\}\{df(x)/dx\}^2.$$

Lemma 2.1 *Let p be a semi p-operator of $\mathcal{F}$. If $f \in \mathcal{F}$ and $0 \notin f(X)$, then we have*

$$p \circ \tilde{f} = -\tilde{f}^2(p \circ f).$$

Proof: By (**viii**) in Definition 2.1, $\tilde{f} \in \mathcal{F}$ exists under the given condition. Then, by (**ii**) in Definition 2.2, we have

$$p \circ (\tilde{f}f) = (p \circ \tilde{f})f + \tilde{f}(p \circ f).$$

By (**i**) in Definition 2.2. the left-hand side of the above equation is zero since $\tilde{f}f$ is a constant function. Q.E.D.

Definition 2.5
(**i**) *If L is a mapping such that $L \circ f \in \mathcal{F}$ for any $f \in \mathcal{F}$, then L is called an operator of $\mathcal{F}$ (Note that any $g \in \mathcal{F}$ is an operator of $\mathcal{F}$ since $gf \in \mathcal{F}$ for any $f \in \mathcal{F}$.);*
(**ii**) *$(Lf) \circ g \triangleq L \circ (fg)$ for any $f, g \in \mathcal{F}$;*
(**iii**) *$(fL) \circ g \triangleq f(L \circ g)$ for any $f, g \in \mathcal{F}$;*
(**iv**) *If an operator L of $\mathcal{F}$ satisfies $L \circ (\alpha f + \beta g) = \alpha L \circ f + \beta L \circ g$ for any $f, g \in \mathcal{F}$ and any scalars α, β, then L is called a linear operator of $\mathcal{F}$.*

Definition 2.6 *Multiplication of two operators L_1 and L_2 of $\mathcal{F}$ is defined by*

$$(L_1 \circ L_2) \circ f \triangleq L_1 \circ (L_2 \circ f)$$

for all $f \in \mathcal{F}$.

Definition 2.7 *Addition and subtraction of two operators L_1 and L_2 of $\mathcal{F}$ are defined by*

$$(L_1 \pm L_2) \circ f \triangleq L_1 \circ f \pm L_2 \circ f$$

for all $f \in \mathcal{F}$.

Remark 2.8 It can be easily proved that a sum of semi p-operators is a semi p-operator.

Definition 2.8 *An operator L of $\mathcal{F}$ is called the zero operator of $\mathcal{F}$ if and only if $L \circ f = 0$ for all $f \in \mathcal{F}$.*

Definition 2.9 *The operators L_1 and L_2 of $\mathcal{F}$ are equal if and only if $L_1 - L_2$ is the zero operator of $\mathcal{F}$.*

Theorem 2.3 *Suppose that the topological space Y is a field. And let $\mathcal{A}$ be the collection of all functions $f : X \to Y$ such that addition and multiplication in $\mathcal{A}$ are induced from those in Y in a natural way:$(f+g)(x) \triangleq f(x) + g(x) = g(x) + f(x)$ and $(fg)(x) \triangleq f(x)g(x) = g(x)f(x)$. If $\mathcal{A}$ has some semi p-operator, then $\mathcal{A}$ is the space $\mathcal{F}$ in Definition 2.1.*

Proof: Since Y is a field, $\mathcal{A}$ satisfies (**i**)-(**viii**) in Definition 2.1 by regarding that Y is also the coefficient field of $\mathcal{A}$. Hence, by Definition 2.1 and by assumption that some semi p-operator exists for $\mathcal{A}$, $\mathcal{A}$ is the space $\mathcal{F}$ in Definition 2.1. Q.E.D.

Index